

## Research Article

# Phase Transitions, Geometrothermodynamics, and Critical Exponents of Black Holes with Conformal Anomaly

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We investigate the phase transitions of black holes with conformal anomaly in canonical ensemble. Some interesting and novel phase transition phenomena have been discovered. It is shown that there are striking differences in both Hawking temperature and phase structure between black holes with conformal anomaly and those without it. Moreover, we probe in detail the dependence of phase transitions on the choice of parameters. The results show that black holes with conformal anomaly have much richer phase structure than those without it. There would be two, only one, or no phase transition points depending on the parameters. The corresponding parameter regions are derived both numerically and graphically. Geometrothermodynamics are built up to examine the phase structure we have discovered. It is shown that Legendre invariant thermodynamic scalar curvature diverges exactly where the specific heat diverges. Furthermore, critical behaviors are investigated by calculating the relevant critical exponents. And we prove that these critical exponents satisfy the thermodynamic scaling laws.

## 1. Introduction

Black hole thermodynamics has long been one of the exciting and challenging research fields ever since the pioneer research made by Bekenstein and Hawking [1, 2]. A variety of thermodynamic quantities of black holes have been studied. In 1983, Hawking and Page [3] discovered that pure thermal radiation in AdS space becomes unstable above certain temperature and collapses to form black holes. This is the well-known Hawking-Page phase transition which describes the phase transition between the Schwarzschild AdS black hole and the thermal AdS space. This phenomenon can be interpreted in the AdS/CFT correspondence [4] as the confinement/deconfinement phase transition of gauge field [5]. Since then, phase transitions of black holes have been investigated from different perspectives. For recent papers, see [6–26].

One of the elegant approaches is the thermodynamic geometry method, which was first introduced by Weinhold [27] and Ruppeiner [28]. Weinhold proposed metric structure in the energy representation as  $g_{i,j}^W =$

$\partial_i \partial_j M(U, N^a)$  while Ruppeiner defined metric structure as  $g_{i,j}^R = -\partial_i \partial_j S(U, N^a)$ . These metric structures are, respectively, the Hessian matrix of the internal energy  $U$  and the entropy  $S$  with respect to the extensive thermodynamic variables  $N^a$ . And Weinhold's metrics were found to be conformally connected to Ruppeiner's metrics through the map  $dS_R^2 = dS_W^2/T$  [29]. Ruppeiner's metric has been applied to investigate various thermodynamics systems for its profound physical meaning. For more details, see the nice review in [30]. Recently, Quevedo [31] presented a new formalism called geometrothermodynamics, which allows us to derive Legendre invariant metrics in the space of equilibrium states. Geometrothermodynamics presents a unified geometry where the metric structure describes various types of black hole thermodynamics [32–41].

Apart from the thermodynamic geometry, critical behavior also plays a crucial role in the study of black hole phase transitions. The critical points of phase transitions are characterized by the discontinuity of thermodynamic quantities. So, it is important to investigate the behavior

in the neighborhood of the critical point, especially the divergences of various thermodynamic quantities. In classical thermodynamics, this goal is achieved by taking into account a set of critical exponents, from which we can gain qualitative insights into the critical behavior. These critical exponents are found to be universal to a large extent (only depending on the dimensionality, symmetry, etc.) and satisfy scaling laws, which can be attributed to scaling hypothesis. Critical behavior of black holes accompanied with their critical exponents has been investigated not only in asymptotically flat space time [42–48] but also in the de Sitter and anti-de Sitter space [49–55].

In this paper, we would like to focus our attention on the critical behavior and geometrothermodynamics of static and spherically symmetric black holes with conformal anomaly. As we know, conformal anomaly, an important concept with a long history, has various applications in quantum field theory in curved spaces, string theory, black hole physics, and cosmology. So it is worth probing its influences in phase transitions of black holes. Recently, Cai et al. [56] found a class of static and spherically symmetric black holes with conformal anomaly, whose thermodynamic quantities were also investigated in the same paper. It was found that there exists a logarithmic correction to the well-known Bekenstein-Hawking area entropy. This discovery is quite important in the sense that with this term one is able to compare black hole entropy up to the subleading order, in the gravity side and in the microscopic statistical interpretation side [56]. Based on the metrics in that paper, phase transitions of a spherically symmetric Schwarzschild black hole have been investigated by taking into account the back reaction through the conformal anomaly of matter fields recently [57]. It has been shown that there exists an additional phase transition to the conventional Hawking-Page phase transition. The entropy of these black holes has also been investigated by using quantum tunneling approach [58]. Moreover, Ehrenfest equation has been applied to investigate this class of black holes [59] and it has been found that the phase transition is a second-order one. Despite these achievements, there are still many issues left to be explored. Reference [57] mainly focuses on the uncharged case. So, it is natural to ask what would happen to the charged black holes. In [59], the authors concentrated their efforts on the Ehrenfest equation in the grand canonical ensemble. So it is worthwhile to study the phase transition in canonical ensemble. The dependence of the phase structure on the parameter deserves to be further investigated. One may also wonder whether the thermodynamic geometry and scaling laws still work to reveal the phase structure and critical behavior when conformal anomaly is taken into consideration. Motivated by these, we would like to investigate the phase transition, geometrothermodynamics, and critical exponents in canonical ensemble.

The organization of our paper is as follows. In Section 2, the thermodynamics of black holes with conformal anomaly will be briefly reviewed. In Section 3, phase transitions will be investigated in detail in canonical ensemble, and some interesting and novel phase transition phenomena will be disclosed. In Section 4, geometrothermodynamics will be established to examine the phase structure we find in

Section 3. In Section 5, critical exponents will be calculated and the scaling laws will be examined. In the end, conclusions will be drawn in Section 6.

## 2. A Brief Review of Thermodynamics

The static and spherically symmetric black hole solution with conformal anomaly has been proposed as [56]

$$ds^2 = f(r) dt^2 - \frac{dr^2}{f(r)} - r^2 (d\theta^2 + \sin^2\theta d\varphi^2), \quad (1)$$

where

$$f(r) = 1 - \frac{r^2}{4\tilde{\alpha}} \left( 1 - \sqrt{1 - \frac{16\tilde{\alpha}M}{r^3} + \frac{8\tilde{\alpha}Q^2}{r^4}} \right). \quad (2)$$

The Newton constant  $G$  has been set to one. Both  $M$  and  $Q$  are integration constants. And the coefficient  $\tilde{\alpha}$  is positive. The physical meanings of  $M$  and  $Q$  were discussed in [56].  $M$  is nothing but the mass of the solution while  $Q$  should be interpreted as  $U(1)$  charge of some conformal field theory.

When  $M = Q = 0$ , the metric above reduces to

$$ds^2 = dt^2 - dr^2 - r^2 (d\theta^2 + \sin^2\theta d\varphi^2), \quad (3)$$

implying that the vacuum limit is the Minkowski space-time.

In the large  $r$  limit, (2) becomes

$$f(r) \approx 1 - \frac{2M}{r} + \frac{Q^2}{r^2} + O(r^{-2}), \quad (4)$$

which behaves like the Reissner-Nordström solution.

When  $\tilde{\alpha} \rightarrow 0$ , (2) reduces into

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}. \quad (5)$$

Equations (1) and (5) consist of the metric of Reissner-Nordström black hole.

Solving the equation  $f(r) = 0$ , we can get the radius of black hole horizon  $r_+$ , with which the mass of the black hole can be expressed as

$$M = \frac{r_+}{2} + \frac{Q^2}{2r_+} - \frac{\tilde{\alpha}}{r_+}. \quad (6)$$

The Hawking temperature can be derived as

$$T = \frac{f'(r_+)}{4\pi} = \frac{r_+^2 + 2\tilde{\alpha} - Q^2}{4\pi r_+ (r_+^2 - 4\tilde{\alpha})}. \quad (7)$$

The potential difference between the horizon and the infinity can be written as

$$\Phi = \frac{Q}{r_+}. \quad (8)$$

The entropy was reviewed in [59] as

$$S = \pi r_+^2 - 4\pi\tilde{\alpha} \ln r_+^2. \quad (9)$$

### 3. Novel Phase Transition Phenomena

In this section, we would like to investigate the phase transition of black holes with conformal anomaly in canonical ensemble where the charge of the black hole is fixed.

The corresponding specific heat can be calculated as

$$C_Q = T \left( \frac{\partial S}{\partial T} \right)_Q = \frac{2\pi(r_+^2 - 4\tilde{\alpha})^2 (Q^2 - 2\tilde{\alpha} - r_+^2)}{r_+^4 - 8\tilde{\alpha}^2 + 10r_+^2\tilde{\alpha} + Q^2(4\tilde{\alpha} - 3r_+^2)}. \quad (10)$$

Apparently,  $C_Q$  may diverge when

$$r_+^4 - 8\tilde{\alpha}^2 + 10r_+^2\tilde{\alpha} + Q^2(4\tilde{\alpha} - 3r_+^2) = 0, \quad (11)$$

which suggests a possible phase transition. However, the phase transition point characterized by (11) is not apparent. To gain an intuitive understanding, we plot Figure 1(a) using (10). To check whether the phase transition point locates in the physical region, we also plot the Hawking temperature using (7) in Figure 1(b). It is shown that the phase transition point locates in the positive temperature region. From Figures 1(a) and 1(b), we find that there have been striking differences between the case  $\tilde{\alpha} \neq 0$  and the case  $\tilde{\alpha} = 0$ . In the case  $Q = 1, \tilde{\alpha} = 0.1$ , there are two phase transition points while there is only one in the case  $\tilde{\alpha} = 0$ . The temperature in the case  $\tilde{\alpha} = 0$  increases monotonically while there exists local minimum temperature in the case  $Q = 1, \tilde{\alpha} = 0.1$ . Figure 1(a) can be divided into four phases. The first one is thermodynamically stable ( $C_Q > 0$ ) with small radius. The second one is unstable ( $C_Q < 0$ ) with medium radius. The third one is thermodynamically stable ( $C_Q > 0$ ) with medium radius while the fourth one is thermodynamically unstable ( $C_Q < 0$ ) with large radius. So the phase transition takes place not only from an unstable large black hole to a locally stable medium black hole but also from an unstable medium black hole to a locally stable small black hole. From Figure 1(b), we notice that the Hawking temperature has a local minimum value. And the corresponding location can be derived through

$$\frac{\partial T}{\partial r_+} = -\frac{r_+^4 - 8\tilde{\alpha}^2 + 10r_+^2\tilde{\alpha} + Q^2(4\tilde{\alpha} - 3r_+^2)}{4\pi(r_+^3 - 4r_+\tilde{\alpha})^2} = 0. \quad (12)$$

It is quite interesting to note that the numerator of (12) is the same as (11), which implies that the location which corresponds to the minimum Hawking temperature also witnesses the existence of phase transition.

To probe the dependence of phase transition location on the choice of parameter, we solve (11) and obtain the location of phase transition point as

$$r_c = \sqrt{\frac{3Q^2 - 10\tilde{\alpha} \pm \sqrt{132\tilde{\alpha}^2 - 76\tilde{\alpha}Q^2 + 9Q^4}}{2}}. \quad (13)$$

With (13) at hand, we plot Figures 2(a) and 2(b) which show the influence of parameters  $Q$  and  $\tilde{\alpha}$ , respectively. It can be observed from Figures 2(a) and 2(b) that black holes with conformal anomaly have much richer phase structure than

that without conformal anomaly. When  $\tilde{\alpha} = 0$ , the location of the phase transition  $r_c$  is proportional to the charge  $Q$ . However, the cases of black holes with conformal anomaly are quite complicated. For  $\tilde{\alpha} = 0.1$ , the curve can be divided into three regions. Through numerical calculation, we find that black holes have only one phase transition point when  $Q < (0, 0.4472)$ . When  $0.4472 < Q < 0.7746$ , there would be no phase transition at all. When  $Q > 0.7746$ , there exist two phase transition points, just as what we show in Figure 1(a). And the distance between these two phase transition points becomes larger with the increasing of  $Q$ . Figure 2(b) shows the case that the charge  $Q$  has been fixed at one. We notice that there would be two phase transition points when  $0 < \tilde{\alpha} < 1/6$ , which is consistent with Figure 1(a). And the distance between these two phase transition points becomes narrower with the increasing of  $\tilde{\alpha}$ . When  $\tilde{\alpha} < (1/6, 1/2)$ , there would be no phase transition. When  $\tilde{\alpha} > 1/2$ , there would be only one phase transition point. To gain a three-dimensional understanding, we also include a three-dimensional figure of  $C_Q$  in Figures 3(a) and 3(b).

Apart from the specific heat, we would also like to investigate the behavior of the inverse of the isothermal compressibility, which is defined as

$$\kappa_T^{-1} = Q \left( \frac{\partial \Phi}{\partial Q} \right)_T. \quad (14)$$

Utilizing the thermodynamic identity relation

$$\left( \frac{\partial \Phi}{\partial T} \right)_Q \left( \frac{\partial T}{\partial Q} \right)_\Phi \left( \frac{\partial Q}{\partial \Phi} \right)_T = -1, \quad (15)$$

we obtain

$$\left( \frac{\partial \Phi}{\partial Q} \right)_T = - \left( \frac{\partial \Phi}{\partial T} \right)_Q \left( \frac{\partial T}{\partial Q} \right)_\Phi, \quad (16)$$

where the second term on the right hand side can be calculated through

$$\left( \frac{\partial T}{\partial Q} \right)_\Phi = \left( \frac{\partial T}{\partial r_+} \right)_Q \left( \frac{\partial r_+}{\partial Q} \right)_\Phi + \left( \frac{\partial T}{\partial Q} \right)_{r_+}. \quad (17)$$

Utilizing (7), (8), (14), (16), and (17), we obtain the explicit form of  $\kappa_T^{-1}$  as

$$\kappa_T^{-1} = \frac{Qr_+^4 - Q^3r_+^2 - 4Q^3\tilde{\alpha} + 10Qr_+^2\tilde{\alpha} - 8Q\tilde{\alpha}^2}{r_+ [r_+^4 - 8\tilde{\alpha}^2 + 10r_+^2\tilde{\alpha} + Q^2(4\tilde{\alpha} - 3r_+^2)]}. \quad (18)$$

We show the behavior of  $\kappa_T^{-1}$  in Figure 4. Comparing Figure 4 with Figure 1(a), we find that the inverse of the isothermal compressibility  $\kappa_T^{-1}$  also diverges at the critical point.

### 4. Geometrothermodynamics

According to geometrothermodynamics [31], the  $(2n + 1)$ -dimensional thermodynamic phase space  $\mathcal{T}$  can be coordinated by the set of independent quantities  $\{\phi, E^a, I^a\}$ , where

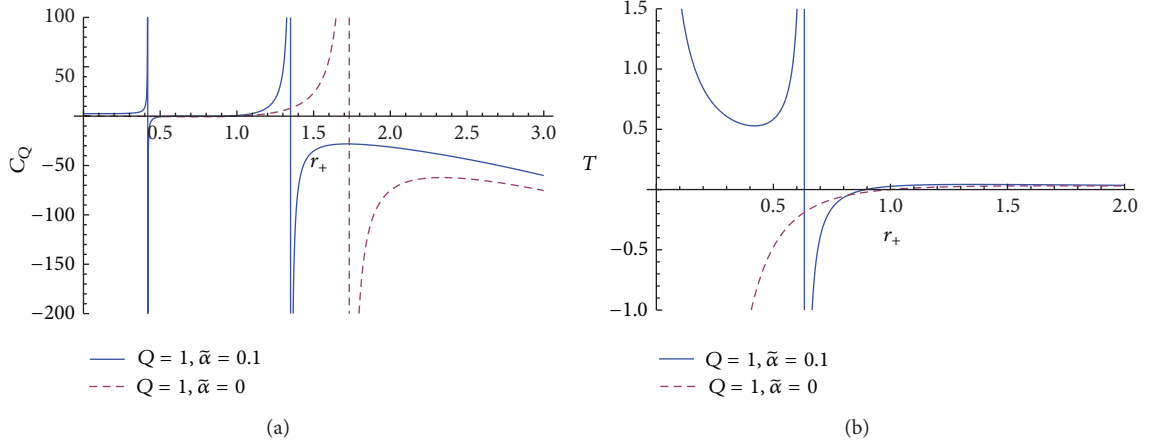


FIGURE 1: (a)  $C_Q$  versus  $r_+$  for  $Q = 1, \tilde{\alpha} = 0.1$ . (b)  $T$  versus  $r_+$  for  $Q = 1, \tilde{\alpha} = 0.1$ .

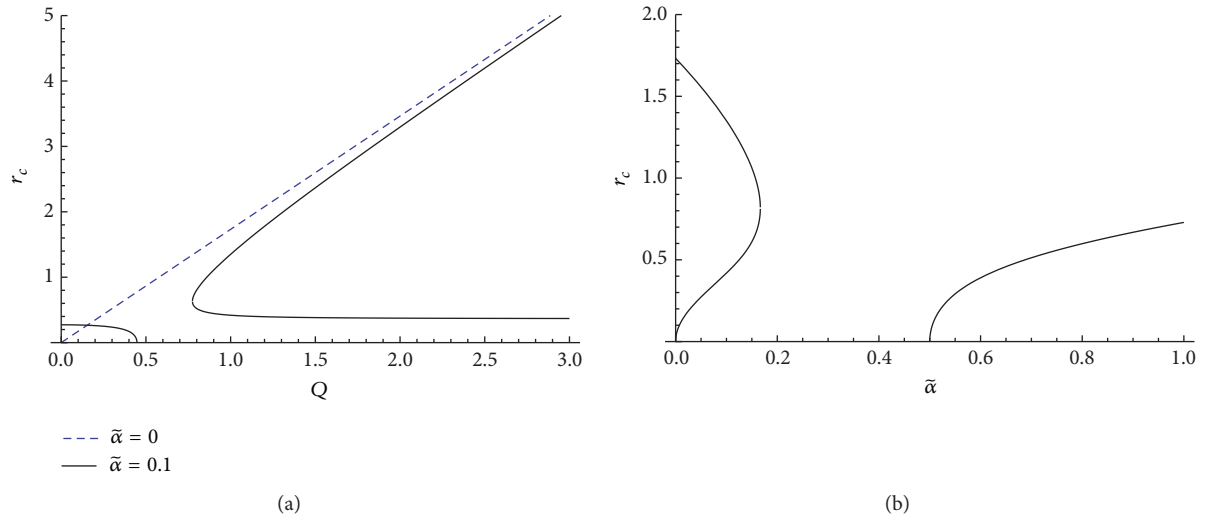


FIGURE 2: (a)  $r_c$  versus  $Q$  for  $\tilde{\alpha} = 0.1$  and  $\tilde{\alpha} = 0$ . (b)  $r_c$  versus  $\tilde{\alpha}$  for  $Q = 1$ .

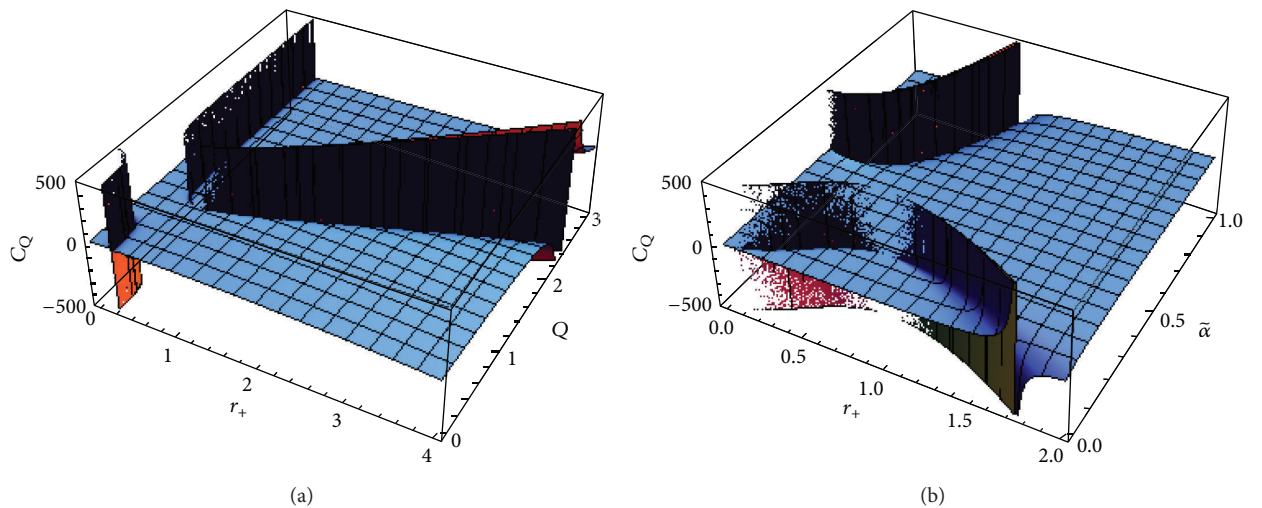


FIGURE 3: (a)  $C_Q$  versus  $Q$  and  $r_+$  for  $\tilde{\alpha} = 0.1$ . (b)  $C_Q$  versus  $\tilde{\alpha}$  and  $r_+$  for  $Q = 1$ .

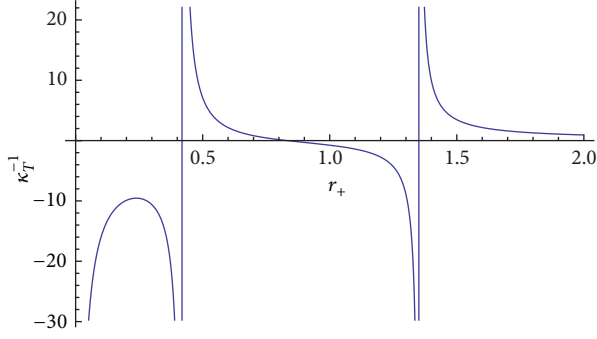


FIGURE 4: The inverse of the isothermal compressibility  $\kappa_T^{-1}$  versus  $r_+$  for  $Q = 1, \tilde{\alpha} = 0.1$ .

$\phi$  corresponds to the thermodynamic potential and  $E^a, I^a$  are the extensive and intensive thermodynamic variables, respectively. The fundamental Gibbs 1-form defined on  $\mathcal{T}$  can then be written as  $\Theta = d\phi - \delta_{ab} I^a dE^b$ , where  $\delta_{ab} = \text{diag}(1, \dots, 1)$ . Considering a nondegenerate Riemannian metric  $G$ , a contact Riemannian manifold can be defined from the set  $(\mathcal{T}, \Theta, G)$  if the condition  $\Theta \wedge (\Theta)^n \neq 0$  is satisfied. Utilizing a smooth map  $\varphi : \varepsilon \rightarrow \mathcal{T}$ , that is,  $\varphi : (E^a) \mapsto (\phi, E^a, I^a)$ , a submanifold  $\varepsilon$  called the space of thermodynamic equilibrium states can be induced. Furthermore, a thermodynamic metric  $g$  can be induced in the equilibrium manifold  $\varepsilon$  by the smooth map  $\varphi$ .

As proposed by Quevedo et al., the nondegenerate metric  $G$  and the thermodynamic metric  $g$  can be written as follows [38]:

$$G = (d\phi - \delta_{ab} I^a dE^b)^2 + (\delta_{ab} E^a I^b) (\eta_{cd} dE^c dI^d), \quad (19)$$

$$g = \varphi^* (G) = \left( E^c \frac{\partial \phi}{\partial E^c} \right) \left( \eta_{ab} \delta^{bc} \frac{\partial^2 \phi}{\partial E^c \partial E^d} dE^a dE^d \right), \quad (20)$$

where  $\eta_{ab} = \text{diag}(-1, \dots, 1)$ .

To construct geometrothermodynamics of black holes with conformal anomaly in canonical ensemble, we choose  $M$  to be the thermodynamic potential and  $S, Q$  to be the extensive variables. Then the corresponding thermodynamic phase space is a 5-dimensional one coordinated by the set of independent coordinates  $\{M, S, Q, T, \Phi\}$ . The fundamental Gibbs 1-form defined on  $\mathcal{T}$  can then be written as

$$\Theta = dM - TdS - \Phi dQ. \quad (21)$$

The nondegenerate metric  $G$  from (19) can be changed into

$$G = (dM - TdS - \Phi dQ)^2 + (TS + \Phi Q) (-dSdT + dQd\Phi). \quad (22)$$

Introducing the map

$$\varphi : \{S, Q\} \mapsto \left\{ M(S, Q), S, Q, \frac{\partial M}{\partial S}, \frac{\partial M}{\partial Q} \right\}, \quad (23)$$

the space of thermodynamic equilibrium states can be induced. According to (19), the thermodynamic metric  $g$  can be written as follows:

$$g = \left( S \frac{\partial M}{\partial S} + Q \frac{\partial M}{\partial Q} \right) \left( -\frac{\partial^2 M}{\partial S^2} dS^2 + \frac{\partial^2 M}{\partial Q^2} dQ^2 \right). \quad (24)$$

Utilizing (6) and (9), we can easily calculate the relevant quantities in (24) as

$$\frac{\partial M}{\partial S} = \frac{r_+^2 + 2\tilde{\alpha} - Q^2}{4\pi r_+ (r_+^2 - 4\tilde{\alpha})}, \quad (25)$$

$$\frac{\partial M}{\partial Q} = \frac{Q}{r_+}, \quad (26)$$

$$\frac{\partial^2 M}{\partial S^2} = \frac{8\tilde{\alpha}^2 - r_+^4 - 10r_+^2\tilde{\alpha} - Q^2(4\tilde{\alpha} - 3r_+^2)}{8\pi^2 r_+ (r_+^2 - 4\tilde{\alpha})^3}, \quad (27)$$

$$\frac{\partial^2 M}{\partial Q^2} = \frac{1}{r_+}. \quad (28)$$

Comparing (25) and (26) with (7) and (8), we find

$$\frac{\partial M}{\partial S} = T, \quad \frac{\partial M}{\partial Q} = \Phi, \quad (29)$$

which proves the validity of the first law of black hole thermodynamics  $dM = TdS + \Phi dQ$ . Substituting (25)–(28) into (24), we can calculate the component of the thermodynamic metric  $g$  as

$$\begin{aligned} g_{SS} &= \frac{1}{32\pi^2 r_+^2 (r_+^2 - 4\tilde{\alpha})^4} \\ &\times [r_+^4 - 8\tilde{\alpha}^2 + 10r_+^2\tilde{\alpha} + Q^2(4\tilde{\alpha} - 3r_+^2)] \\ &\times [r_+^4 + 2r_+^2\tilde{\alpha} + Q^2(3r_+^2 - 16\tilde{\alpha}) + 8\tilde{\alpha}(Q^2 - r_+^2 - 2\tilde{\alpha}) \ln r_+], \\ g_{QQ} &= \frac{r_+^4 - 16Q^2\tilde{\alpha} + 2r_+^2\tilde{\alpha} + 3Q^2r_+^2 + (8Q^2\tilde{\alpha} - 8r_+^2\tilde{\alpha} - 16\tilde{\alpha}^2) \ln r_+}{4r_+^2 (r_+^2 - 4\tilde{\alpha})}. \end{aligned} \quad (30)$$

Utilizing (30), we can obtain the Legendre invariant scalar curvature as

$$\mathfrak{R}_Q = \frac{A(x_+, Q)}{B(x_+, Q)}, \quad (31)$$

where

$$\begin{aligned} B(x_+, Q) &= [r_+^4 + 10r_+^2\tilde{\alpha} - 8\tilde{\alpha}^2 + Q^2(4\tilde{\alpha} - 3r_+^2)]^2 \\ &\times [r_+^4 + Q^2(3r_+^2 - 16\tilde{\alpha}) + 2r_+^2\tilde{\alpha} + 8\tilde{\alpha}(Q^2 - r_+^2 - 2\tilde{\alpha}) \ln r_+]^3. \end{aligned} \quad (32)$$



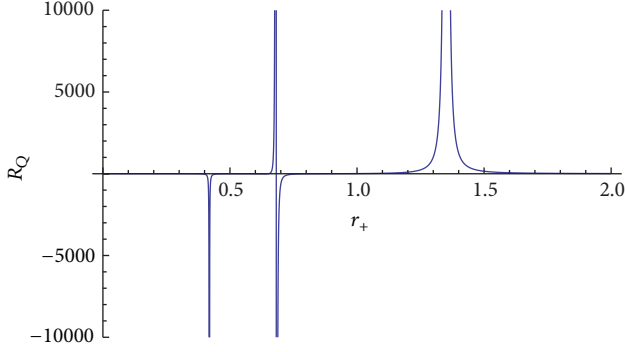


FIGURE 5: Thermodynamic scalar curvature  $R_Q$  versus  $r_+$  for  $Q = 1, \tilde{\alpha} = 0.1$ .

The numerator of the Legendre invariant scalar curvature is too lengthy to be displayed here. From (32), we find that the Legendre invariant scalar curvature shares the same factor  $r_+^4 + 10r_+^2\tilde{\alpha} - 8\tilde{\alpha}^2 + Q^2(4\tilde{\alpha} - 3r_+^2)$  with the specific heat  $C_Q$  in its denominator, which implies that it would diverge when  $r_+^4 + 10r_+^2\tilde{\alpha} - 8\tilde{\alpha}^2 + Q^2(4\tilde{\alpha} - 3r_+^2) = 0$ . That is the exact condition that the phase transition point satisfies. To get an intuitive sense on this issue, we plot Figure 5 showing the behavior of thermodynamic scalar curvature  $\mathfrak{R}_Q$ . From Figure 5, we find that thermodynamic scalar curvature  $\mathfrak{R}_Q$  diverges at three locations. Comparing Figure 5 with Figure 1(b), we find that the second diverging point which corresponds to negative Hawking temperature does not have physical meaning. Furthermore, the first and the third diverging points coincide exactly with the phase transition point, which can be witnessed by comparing Figure 5 with Figure 1(a). So we can safely draw the conclusion that the Legendre invariant metric constructed in geometrothermodynamics correctly produces the behavior of the thermodynamic interaction and phase transition structure of black holes with conformal anomaly.

## 5. Critical Exponents and Scaling Laws

In order to have a better understanding of the phase transition of black holes with conformal anomaly, we would like to investigate their critical behavior near the critical point by considering a set of critical exponents in this section.

Before embarking on calculating critical exponents, we would like to reexpress physical quantities near the critical point as

$$r_+ = r_c (1 + \Delta), \quad (33)$$

$$T(r_+) = T_c (1 + \varepsilon), \quad (34)$$

$$Q(r_+) = Q_c (1 + \eta), \quad (35)$$

where  $|\Delta| \ll 1, |\varepsilon| \ll 1, |\eta| \ll 1$ . Note that the subindex “c” in this section denotes the value of the physical quantity (or the expression) at the critical point. For example,  $T_c$  corresponds to the temperature at the critical point.

Critical exponent  $\alpha$  is defined through

$$C_Q \sim |T - T_c|^{-\alpha}. \quad (36)$$

To obtain  $T - T_c$ , we would like to carry out Taylor expansion as below:

$$\begin{aligned} T(r_+) &= T_c + \left[ \left( \frac{\partial T}{\partial r_+} \right)_{Q=Q_c} \right]_{r_+=r_c} (r_+ - r_c) \\ &+ \frac{1}{2} \left[ \left( \frac{\partial^2 T}{\partial r_+^2} \right)_{Q=Q_c} \right]_{r_+=r_c} (r_+ - r_c)^2 \\ &+ \text{higher order terms,} \end{aligned} \quad (37)$$

from which we obtain

$$\Delta = \frac{1}{r_c} \sqrt{\frac{2\varepsilon T_c}{D}}, \quad (38)$$

where

$$\begin{aligned} D &= \left[ \left( \frac{\partial^2 T}{\partial r_+^2} \right)_{Q=Q_c} \right]_{r_+=r_c} \\ &= \frac{r_c^6 + 24r_c^4\tilde{\alpha} - 24r_c^2\tilde{\alpha}^2 + 32\tilde{\alpha}^3 - 2Q_c^2(3r_c^4 - 6r_c^2\tilde{\alpha} + 8\tilde{\alpha}^2)}{2\pi(r_c^3 - 4r_c\tilde{\alpha})^3}. \end{aligned} \quad (39)$$

In the above derivation, we have considered the fact that  $C_Q$  diverges at the critical point, making the second term at the right hand side of (37) vanish. Substituting (33) into (10) and keeping only the linear terms in its denominator, we obtain

$$C_Q \simeq \frac{2\pi(r_c^2 - 4\tilde{\alpha})^2 (Q_c^2 - 2\tilde{\alpha} - r_c^2)}{\Delta(4r_c^4 + 20r_c^2\tilde{\alpha} - 6Q_c^2r_c^2)}, \quad (40)$$

which can be transformed via (38) into

$$C_Q \simeq \frac{\pi\sqrt{2D}(r_c^2 - 4\tilde{\alpha})^2 (Q_c^2 - 2\tilde{\alpha} - r_c^2)}{(4r_c^3 + 20r_c\tilde{\alpha} - 6Q_c^2r_c)(T - T_c)^{1/2}}. \quad (41)$$

Comparing (41) with (36), we can obtain  $\alpha = 1/2$ .

Critical exponent  $\beta$  is defined through the following relation when  $Q$  is fixed:

$$\Phi(r_+) - \Phi(r_c) \sim |T - T_c|^\beta. \quad (42)$$

The above definition motivates us to carry out the Taylor expansion as

$$\begin{aligned} \Phi(r_+) &= \Phi_c + \left[ \left( \frac{\partial \Phi}{\partial r_+} \right)_{Q=Q_c} \right]_{r_+=r_c} (r_+ - r_c) \\ &+ \text{higher order terms.} \end{aligned} \quad (43)$$

Utilizing (8) and (43) and neglecting higher order terms of (42), we get

$$\begin{aligned}\Phi(r_+) - \Phi_c &= \left[ \left( \frac{\partial \Phi}{\partial r_+} \right)_{Q=Q_c} \right]_{r_+=r_c} \sqrt{\frac{2}{D}} (T - T_c)^{1/2} \\ &= -\frac{Q_c}{r_c^2} \sqrt{\frac{2}{D}} (T - T_c)^{1/2}.\end{aligned}\quad (44)$$

Comparing (42) with (44), we can obtain  $\beta = 1/2$ .

Critical exponent  $\gamma$  is defined through the following relation:

$$\kappa_T^{-1} \sim |T - T_c|^{-\gamma}.\quad (45)$$

Substituting (33) and (38) into (18) and keeping only the linear term of  $\Delta$ , we obtain

$$\kappa_T^{-1} = \frac{\sqrt{D} (Q_c r_c^4 - Q_c^3 r_c^2 - 4Q_c^3 \tilde{\alpha} + 10Q_c r_c^2 \tilde{\alpha} - 8Q_c \tilde{\alpha}^2)}{\sqrt{2} [5r_c^4 - 8\tilde{\alpha}^2 + 30r_c^2 \tilde{\alpha} + Q_c^2 (4\tilde{\alpha} - 9r_c^2)] (T - T_c)^{1/2}}.\quad (46)$$

From (45) and (46), we find that  $\gamma = 1/2$ .

Critical exponent  $\delta$  is defined for the fixed temperature  $T_c$  through

$$\Phi(r_+) - \Phi(r_c) \sim |Q - Q_c|^{1/\delta}.\quad (47)$$

To obtain  $Q - Q_c$ , we would like to carry out Taylor expansion as

$$\begin{aligned}Q(r_+) &= Q_c + \left[ \left( \frac{\partial Q}{\partial r_+} \right)_T \right]_{r_+=r_c} (r_+ - r_c) \\ &\quad + \frac{1}{2} \left[ \left( \frac{\partial^2 Q}{\partial r_+^2} \right)_T \right]_{r_+=r_c} (r_+ - r_c)^2 \\ &\quad + \text{higher order terms}.\end{aligned}\quad (48)$$

Utilizing the thermodynamic identity again, we get

$$\left[ \left( \frac{\partial Q}{\partial r_+} \right)_T \right]_{r_+=r_c} = - \left[ \left( \frac{\partial T}{\partial r_+} \right)_Q \right]_{r_+=r_c} \left[ \left( \frac{\partial Q}{\partial T} \right)_{r_+} \right]_{r_+=r_c} = 0.\quad (49)$$

In the above derivation, we have taken into account the fact that  $C_Q$  diverges at the critical point, making the first term at the right hand side of (37) vanish. Substituting (33) and (35) into (48) and neglecting the high order terms, we obtain

$$\Delta = \sqrt{\frac{2Q_c \eta}{Er_c^2}},\quad (50)$$

where

$$\begin{aligned}E &= \left[ \left( \frac{\partial^2 Q}{\partial r_+^2} \right)_T \right]_{r_+=r_c} \\ &= \frac{22r_c^4 \tilde{\alpha} + 32\tilde{\alpha}^3 + 16\tilde{\alpha}^2 (r_c^2 - Q_c^2) - r_c^4 (3Q_c^2 + r_c^2)}{2Q_c (r_c^3 - 4r_c \tilde{\alpha})^2}.\end{aligned}\quad (51)$$

Performing Taylor expansion of the function  $\Phi$  near the critical point, we get

$$\begin{aligned}\Phi(r_+) &= \Phi_c + \left[ \left( \frac{\partial \Phi}{\partial r_+} \right)_T \right]_{r_+=r_c} (r_+ - r_c) \\ &\quad + \text{higher order terms},\end{aligned}\quad (52)$$

where the coefficient of the second term on the right hand side can be derived as follows:

$$\begin{aligned}\left[ \left( \frac{\partial \Phi}{\partial r_+} \right)_T \right]_{r_+=r_c} &= \left[ \left( \frac{\partial Q}{\partial r_+} \right)_T \right]_{r_+=r_c} \left[ \left( \frac{\partial \Phi}{\partial Q} \right)_{r_+} \right]_{r_+=r_c} \\ &\quad + \left[ \left( \frac{\partial \Phi}{\partial r_+} \right)_Q \right]_{r_+=r_c} = -\frac{Q_c}{r_c^2}.\end{aligned}\quad (53)$$

Utilizing (50), (52), and (53), we get

$$\Phi(r_+) - \Phi_c \simeq -\frac{Q_c}{r_c^2} \sqrt{\frac{2(Q - Q_c)}{E}},\quad (54)$$

from which we can draw the conclusion that  $\delta = 2$ .

Critical exponent  $\varphi$  is defined through

$$C_Q \sim |Q - Q_c|^{-\varphi}.\quad (55)$$

Substituting (50) into (40), we obtain

$$C_Q \simeq \frac{\pi r_c \sqrt{2E} (r_c^2 - 4\tilde{\alpha})^2 (Q_c^2 - 2\tilde{\alpha} - r_c^2)}{\sqrt{Q - Q_c} (4r_c^4 + 20r_c^2 \tilde{\alpha} - 6Q_c^2 r_c^2)}.\quad (56)$$

Comparing (56) and (55), we find that  $\varphi = 1/2$ .

Critical exponent  $\psi$  is defined through

$$S(r_+) - S_c \sim |Q - Q_c|^\psi.\quad (57)$$

Performing Taylor expansion of the function  $S$  near the critical point, we obtain

$$S(r_+) = S_c + \left[ \left( \frac{\partial S}{\partial r_+} \right)_Q \right]_{r_+=r_c} (r_+ - r_c) + \text{higher order terms}.\quad (58)$$

Utilizing (9), (33), (50), and (58), we get

$$S(r_+) - S_c \simeq \left( 2\pi r_c - \frac{8\pi \tilde{\alpha}}{r_c} \right) \sqrt{\frac{2(Q - Q_c)}{E}},\quad (59)$$

from which we obtain  $\psi = 1/2$ .

Till now, we have finished the calculations of six critical exponents. They are also equal to  $1/2$  except  $\delta = 2$ . Our results are in accordance with those in classical thermodynamics. And it can be easily proved that critical exponents we obtain in our paper satisfy the following thermodynamic scaling laws:

$$\begin{aligned}\alpha + 2\beta + \gamma &= 2, & \alpha + \beta(\delta + 1) &= 2, \\ (2 - \alpha)(\delta\psi - 1) + 1 &= (1 - \alpha)\delta, \\ \gamma(\delta + 1) &= (2 - \alpha)(\delta - 1), & \gamma &= \beta(\delta - 1), \\ \varphi + 2\psi - \delta^{-1} &= 1.\end{aligned}\quad (60)$$

## 6. Conclusions

The phase transition of black holes with conformal anomaly has been investigated in canonical ensemble. Firstly, we calculate the relevant thermodynamic quantities and discuss the behavior of the specific heat at constant charge. We find that there have been striking differences between black holes with conformal anomaly and those without conformal anomaly. In the case  $Q = 1, \tilde{\alpha} = 0.1$ , there are two phase transition points while there is only one in the case  $\tilde{\alpha} = 0$ . The temperature in the case  $\tilde{\alpha} = 0$  increases monotonically while there exists local minimum temperature in the case  $Q = 1, \tilde{\alpha} = 0.1$ . This local minimum temperature corresponds to the phase transition point. We also find that the phase transitions of black holes with conformal anomaly take place not only from an unstable large black hole to a locally stable medium black hole but also from an unstable medium black hole to a locally stable small black hole. We also study the behavior of the inverse of the isothermal compressibility  $\kappa_T^{-1}$  and find that  $\kappa_T^{-1}$  also diverges at the critical point.

Secondly, we probe the dependence of phase transitions on the choice of parameters. The results show that black holes with conformal anomaly have much richer phase structure than that without conformal anomaly. When  $\tilde{\alpha} = 0$ , the location of the phase transition  $r_c$  is proportional to the charge  $Q$ . By contrast, the case of black holes with conformal anomaly is more complicated. For  $\tilde{\alpha} = 0.1$ , the curve can be divided into three regions. Through numerical calculation, we find that black holes have only one phase transition point when  $Q < (0, 0.4472)$ . When  $0.4472 < Q < 0.7746$ , there would be no phase transition at all. When  $Q > 0.7746$ , there exist two phase transition points. And the distance between these two phase transition points becomes larger with the increasing of  $Q$ . In the case that the charge  $Q$  has been fixed at one, we notice that there would be two phase transition points when  $0 < \tilde{\alpha} < 1/6$ . And the distance between these two phase transition points becomes narrower with the increasing of  $\tilde{\alpha}$ . When  $\tilde{\alpha} \in (1/6, 1/2)$ , there would be no phase transition. When  $\tilde{\alpha} > 1/2$ , there would be only one phase transition point.

Thirdly, we build up geometrothermodynamics in canonical ensembles. We choose  $M$  to be the thermodynamic potential and build up both thermodynamic phase space and the space of thermodynamic equilibrium states. We also calculate the Legendre invariant thermodynamic scalar curvature and depict its behavior graphically. It is shown that Legendre invariant thermodynamic scalar curvature diverges exactly where the specific heat diverges. Based on this, we can safely conclude that the Legendre invariant metrics constructed in geometrothermodynamics can correctly produce the behavior of the thermodynamic interaction and phase transition structure even when conformal anomaly is taken into account.

Furthermore, we calculate the relevant critical exponents. They are also equal to  $1/2$  except  $\delta = 2$ . Our results are in accordance with those of other black holes. And it has been proved that critical exponents we obtain in our paper satisfy the thermodynamic scaling laws. We conclude that the critical exponents and the scaling laws do not change even

when we consider conformal anomaly. This may be attributed to the mean field theory.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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